Regularized Approach in 3D Helical Computed Tomography

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Abstract— Helical tomography yields less invasive examination at the expense of degradations in the precision of the reconstructions and the possible presence of specific artifacts. This study presents a new reconstruction method that produces significant enhancement in the precision of helical tomographic images at a reasonable computer cost.

Keywords— Helical tomography, penalized least-square, 3D reconstruction algorithm.

I. INTRODUCTION

X-ray computed tomography (CT) is a fast and mildly invasive technique that produces three-dimensional (3D) images from a set of projections. In planar geometry, the volume is produced by stacking a series of slices reconstructed from two-dimensional (2D) data. Although the precision of such a reconstruction technique is often sufficient for safe medical diagnosis, it remains insufficient to achieve accurate quantitative measurements \cite{1}. In the past ten years, planar scanners have been massively replaced by helical scanners that reduce the scanning time as well as the radiation dose sent to the patient. However, the price to pay for such gains is a loss in the accuracy of the images and the possible presence of strong artifacts. These effects result in part from \textit{ad hoc} interpolation procedures\textsuperscript{1} introduced in order to use the standard planar convolution backprojection (CBP) reconstruction algorithm.

This paper presents a new reconstruction method based on an algebraic formulation of helical tomography in its natural 3D setting. A regularization framework is used to deal with the well-known ill-posed nature of the reconstruction problem. This approach brings significant enhancement to the precision of helical tomographic reconstruction at a reasonable computer cost.

II. METHODOLOGY

We address the problem in a discrete framework; let $f \in \mathbb{R}^N$ and $p_{k} \in \mathbb{R}^{N_{\text{axial}}}$ respectively denote the original 3D scene and the raw data vector that contains the whole set of projections in helical geometry. Our goal is to compute an estimate $\hat{f}$ of $f$. Our approach to reconstruction relies on the minimization of a penalized least-square criterion:

$$J(f) = \|\mathcal{H}(f) - p_{k}\|^2 + \lambda \Phi(f), \quad \lambda \geq 0,$$

where $\Phi$ corresponds to a prior model. Parameter $\lambda$ weights the two terms of the criterion. Difficulties encountered in minimizing (1) are deeply connected to the specificities of models $\mathcal{H}$ and $\Phi$. In the next section, it is shown that under standard assumptions, $\mathcal{H}$ can be given a simple structural form.

A. Model in helical tomography

In planar geometry, it is commonly assumed that the projection data are linked to the object through a sparse linear operator $W$ which approximates a 2D discrete Radon transform \cite{2}. Then, a given projection at angle $\theta_{i}$ corresponds to a submatrix $W_{i}$ extracted from $W$. We now extend this standard projection model to 3D spiral geometry.

Let $f = \{f^{k} \in \mathbb{R}^{K}\}_{k=1}^{K}$ and $(XYZ)$ respectively denote the 3D scene with $K$ voxel slices and an axis system where $OZ$ is the axis of the scanner. $\theta_{i} \in \mathbb{R}$ is a projection angle on the helix. We assume without loss of generality that the axial ray width $\delta$ is not greater than the slice width. As a result, a projection from angle $\theta_{i}$ in our model interacts with at most two slices, say $f^{k}$ and $f^{k+1}$, of respective proportions $\gamma_{k}^{\delta} \in (0; 1] \text{ and } 1 - \gamma_{k}^{\delta} = \gamma_{k+1}^{\delta}$; see Fig. 1-(left). Let $p^{k}$ gather the $N_{\text{axial}}$ projections associated with slices $(f^{k}, f^{k+1})$, and $\theta_{i} \equiv \{\theta_{i}^{k} \in [0; 2\pi]\}_{k=1}^{K}$ be the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Model in helical geometry: YOZ (left), XOY (right).}
\end{figure}

As projections on the helix are interpolated in order to compose "pseudo sets" of projections in pre-determined axial planes.
corresponding set of projection angles. We get,
\[ p^k = H^k \left[ f_{k+1}^k \right] \text{ with } H^k = \begin{bmatrix} W_x W_y & \cdots & W_y W_x \\ \vdots & \ddots & \vdots \\ W_x W_y & \cdots & W_y W_x \end{bmatrix} \in \mathbb{R}^{N \times N}. \] (2)

where \( W_x^k \in \mathbb{R}^{K \times K} \) is the projection operator in planar geometry associated to angle \( \theta_x^k \). Then, (2) yields a linear model for helical tomography: \( p^k = \mathcal{H}(f) = H f \), with
\[ p^k = \begin{bmatrix} p^1 \\ \vdots \\ p^K \end{bmatrix}, \quad H = \begin{bmatrix} [H_x]_d & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & [H_x]_d \end{bmatrix} \in \mathbb{R}^{N \times N}. \] (3)

Compared to planar geometry, helical geometry yields an intrinsically 3D model (L columns overlap between two adjacent blocks, each \( H^k \) being distinct in general). The resulting matrix \( H \) is huge (typically \( 10^{15} \) entries), and storing such a matrix would require an enormous memory capacity in spite of its strong sparsity. Here, we propose to circumvent this problem by assuming that the helix pitch is an integer multiple \( P \) of the slice width. Such an assumption can be met without loss of generality through appropriate axial sampling of the volume. Then, only \( P \) matrices \( H^k \) will suffice to describe \( H \), where \( K/P \) is about the number of helix turns.

**B. \( L_2 L_1 \) regularization**

The image model \( \Phi \) should convey relevant information about \( f \) without jeopardizing the computational feasibility of the minimization of \( J \) (1). We choose \( \Phi \) so as to apply a \( L_2 L_1 \) penalty to the differences in intensity values of neighboring voxels. Such models are well suited to the representation of objects composed of homogeneous areas separated by sharp discontinuities [1]. More specifically, we set \( \Phi(f) = \sum_c \phi(u_c; s) \), where \( \phi(u_c; s) = \sqrt{u_c^2 + s^2} \) and \( s > 0 \) is a free scale parameter, and \( u_c \) \((c = 1, \ldots, M)\) is the difference between a pair of adjacent voxels. We now show that such a model can yield a tractable image reconstruction method.

**III. INVERSION \& RESULTS**

Our choices yield the following expression of \( J \):
\[ J(f) = \| p_k - H f \|^2 + \lambda \sum_{c=1}^{M} \sqrt{u_c^2 + s^2}. \] (4)

Although the minimizer of (4) cannot be expressed in closed form, the context remains favorable since \( J \) is \( C^1 \), convex and coercive. However, the huge size of this minimization problem \((\geq 10^9 \text{ variables})\) precludes the use of standard methods (such as quasi-Newton), and derivation of an efficient algorithm with low numerical count is required. This can be achieved in the same manner as in [1], where a Single Site Update strategy is combined with the structural simplicity of the half-quadratic reformulation of \( J \). The resulting algorithm has a very low computing cost and fast convergence. The performance of the method is illustrated in Fig. 2: A 3D synthetic phantom \((127 \times 127 \times 40 \text{ voxels})\) was used to produce helical noisy (additive Gaussian white noise with zero mean and SNR=26 dB) projection data. Then "standard" reconstruction [3] based on half-scan interpolation and CBP, and \( L_2 L_1 \) reconstruction were performed. Our results show significant improvements when the latter is used.

Estimating only a few slices of the volume or speeding up the reconstruction process is of great interest. Provided one accepts a moderate loss of accuracy, this can be done efficiently by successive reduced regularized reconstructions. The following approximation \( p^k_0 \approx H^k f^k \) based on (2) and (3) leads to the reduced regularized solution \( f^k \) defined as the minimizer of
\[ \tilde{J}(f^k) = \| p^k_0 - H^k f^k \|^2 + \lambda \Phi(f^k) \text{ with } f^k = \begin{bmatrix} f^k_1 \\ \vdots \\ f^k_{K-1} \end{bmatrix} \] (5)

where \( p^k_0 \) and \( H^k \) respectively gather the projections directly involved in slice \( k \) (i.e. \( p^{k-1}_0 \) and \( p^k_0 \) and the three blocks \((H^k)^{k+1} \text{ under a structure similar to (3). Solving (5) yields } f^k \text{ which is a close approximation to } \tilde{f}^k \text{ the } k\text{-th slice of } \tilde{f}. \text{ This technique decreases the computing cost by a factor of } K/2 \text{ while the reconstruction remains accurate, as illustrated in Fig. 2.}

**REFERENCES**